Quantum collapse of dust shells in 2+1 gravity

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Abstract

This paper considers the quantum collapse of infinitesimally thin dust shells in 2 + 1 gravity. In 2 + 1 gravity a shell is no longer a sphere, but a ring of matter. The classical equation of motion of such shells in terms of variables defined on the shell has been considered by Peleg and Steif [1], using the 2 + 1 version of the original formulation of Israel [2], and Crisóstomo and Olea [3], using canonical methods. The minisuperspace quantum problem can be reduced to that of a harmonic oscillator in terms of the curvature radius of the shell, which allows us to use well-known methods to find the motion of coherent wave packets that give the quantum collapse of the shell. Classically, as the radius of the shell falls below a certain point, a horizon forms. In the quantum problem one can define various quantities that give "indications" of horizon formation. Without a proper definition of a "horizon" in quantum gravity, these can be nothing but indications.

It is a great pleasure to be able to contribute to this special volume in honor of Octavio Obregon. One of us [M.R.] has known Octavio for more than 30 years and has collaborated with him on many fruitful projects. We both want to wish him many more productive years and continued success.

1 Introduction

This article describes a toy model of a toy model. There has been some interest over the years in minisuperspace quantization of thin shells as models of the full quantum collapse of more complicated objects. One of most the vexing problems in this scenario has been whether an event horizon will form and consequently some sort of "quantum black hole," or whether a shell of non-interacting parti-

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cles will simply collapse to a point where the uncertainty principle will provide a "repulsive force" and the shell will reexpand into the same universe.

Simple considerations can answer this question up to a point. It seems obvious that for large enough masses one might expect a reasonably peaked wave function centered on a radius near to but outside the classical Schwarzschild radius of the shell would, as it moves toward zero radius, maintain enough coherence to pass beyond the Schwarzschild radius almost in its entirety and a "black hole" would form with high probability. For small masses one might expect that rapid spreading would overwhelm the coherence of the wave function and the shell would reexpand into our own universe with probability essentially (or exactly) one.

In full relativistic quantum gravity these simple ideas are fraught with difficulties. The most serious of these are:

1) The quantum problem becomes unphysical for masses much above the Planck mass. While the minisuperspace approach has frozen out all radiative modes of the gravitational field and one is insisting on a single-particle interpretation of the shell problem, the wave function in the Schrödinger picture can still become pathological at a point where one might expect graviton production to begin. Hájíček [4] has given a sufficient but not necessary condition that shows that we might expect problems above a couple of Planck masses. In [5], a qualitative argument was presented that used Compton wavelength considerations to give a similar bound.

2) There are very serious technical problems in the formulation of the problem. Many authors have used the full ADM method to construct a Hamiltonian for the system, where they have chosen an internal time for the system in order to have a true Hamiltonian and a Schrödinger equation that allows the study of the time evolution of the quantum system. The choice of an internal time leads to the problem of quantum formulations that are not unitarily equivalent. Another problem with these Hamiltonian formulations is that they often require an *ad hoc* choice of a Hamiltonian in terms of variables defined on the shell itself. Several of these Hamiltonians have been given by various authors [4], [6]. A Hamiltonian due to Hájíček and Kuchař [7] has the advantage of being defined by a coherent procedure with no *ad hoc* choices, and is formulated in terms of foliations of spacetime by timelike surfaces. I will discuss this Hamiltonian in more detail below. All of these Hamiltonians have limits on the mass of the shell of a few Planck masses. One formulation that does not seem to have a mass limit is based on the Wheeler-DeWitt equation corresponding to one of the Hamiltonians in [6].

3) Many of the Hamiltonians are quite complicated and there is no real chance of finding analytic solutions to the Schrödinger equations of these models. Numerical solutions have been presented by a group consisting of A. Corichi, G. Cruz, A. Minzoni. M. Rosenbaum and M. Ryan of the UNAM, N. Smyth of the University of Edinburgh, and T. Vukasinac of the University of Michoacan [5].

In order to study a simpler problem (another toy model of a toy model), Ryan [8] considered the quantum collapse of a shell in Newtonian gravity. It is possible to derive a classical equation of motion for the radius of the shell in terms of just the radius, $R(T)$, and the shell mass, M. This problem has the advantages that there is only one time and that the equation is the same as that of a particle falling radially toward a point particle of mass M located at $r = 0$. It is easy to quantize this system, and the final Schrödinger equation is the analogue of the s-state hydrogen-atom equation. While exact analytic solutions exist, one has to use scattering states, and the integrals needed to form wave packets are not tabulated, so only approximate solutions were given in [8].

The idea of the present article is to study the quantum collapse problem in 2 + 1 gravity, where one can address some of the difficulties mentioned above in a context where some of the problems mentioned above do not exist. The quantum problem is unambiguous for all masses, so there is no problem of wave function pathologies. As will be shown below, one possible Hamiltonian for this problem has the form of that of a harmonic oscillator. This will allow us to find simple analytic solutions that can be used to illustrate the development of the quantum collapse of the shell, and will allow us to investigate the problem of horizon formation in terms of analytic constructs. Our toy model of a toy model will help us to understand exactly what is happening in the numerical solutions that will be presented in [9]. In the final section of the article we will mention possible extensions of the 2 + 1 problem that might be the subject of future research.

The plan of the rest of the article is as follows. Section 2 will be a brief resumé of the literature on the general relativistic minisuperspace problem. Section 3 will set up the classical 2 + 1 problem, Section 4 will consider the quantum problem and present solutions, while Section 5 will be conclusions and suggestions for future study.

2 The collapse of thin shells in relativistic quantum gravity

The study of the quantum collapse of dust shells is about a decade old [10]. The classical collapse problem is fairly straightforward, and can in principle be solved exactly. One assumes a δ-function massive (or null) shell where Birkhoff's theorem tells us that outside the shell the metric is Schwarzschild and the metric inside the shell is Minkowski. The Israel junction conditions can be used to derive the equation for the evolution of the shell in terms of intrinsic variables on the shell itself, the proper time, τ, of an observer riding on the shell and the curvature radius, $R(\tau)$, of the shell that he would measure. The equation for the motion of the shell becomes [2]

$$M = \mathcal{M}\left\{1 + \left(\frac{dR}{d\tau}\right)^2\right\}^{1/2} - \frac{\mathcal{M}^2}{2R}, \tag{1}$$

where $\mathcal{M}$ is the rest mass of the shell (a constant of motion) and M is the Schwarzschild mass of the exterior metric. It is straightforward to define $x =$

$\mathcal{M}R$, $V \equiv dx/d\tau$ and $M/\mathcal{M}$ as our Hamiltonian. Assuming $V = V(P)$ and solving $\partial H/\partial P = V(P)$ for the "momentum" P, we find that $V = \sinh^{-1}(P)$ and our Hamiltonian becomes

$$H = \cosh P - \frac{m}{2x}, \tag{2}$$

where $m = \mathcal{M}/M_{pl}$, M_{pl} the Planck mass, a Hamiltonian given by Hájíček [4]. Unfortunately, this is not the only Hamiltonian that gives the equation of motion (1), and we are left with the problem of defining an "appropriate" Hamiltonian for the problem. A number of Hamiltonians for different choices time, including the time of an observer at the center of the shell where space is flat, and a Wheeler-DeWitt equation identical to that for a relativistic charged particle radially falling in a Coulomb potential are given by Hájíček, Kay and Kuchař [6].

Kuchař and Hájíček [7], dissatisfied with such *ad hoc* Hamiltonians, have managed to construct a Hamiltonian for collapsing dust shells that comes directly from an ADM reduction of the Hilbert-plus-matter action. The problem with this approach is that, as spacetime quantities, the matter variables are proportional to $\delta[R - R_0(\tau)]$, where $R_0(\tau)$ is the position of the shell as a function of shell proper time. It is virtually impossible to reduce the time derivatives of these delta functions to reasonable variables in the matter Lagrangian that can give a shell Hamiltonian that describes the motion purely in terms of canonical variables on the shell. Kuchař and Hájíček used an ingenious method based on the fact that the ADM reduction by a $3+1$ foliation is not restricted to foliation by spacelike surfaces, but works just as well for foliations by timelike surfaces. Using this approach and a formulation of the dust fluid velocity in terms of velocity potentials, they define a new Hamiltonian. The cost of this consistent formulation is a very complicated Hamiltonian,

$$H = -\sqrt{2}R\left(1 - \frac{M}{R} - \sqrt{1 - \frac{2M}{R}}\cosh\frac{P}{R}\right)^{1/2} \qquad R \geq 2M, \tag{3}$$

$$H = -\sqrt{2}R\left(1 - \frac{M}{R} - \sqrt{\frac{2M}{R} - 1}\sinh\frac{P}{R}\right)^{1/2} \qquad 0 \leq R \leq 2M. \tag{4}$$

Almost all of the Hamiltonians that have been given (except for the Wheeler-DeWitt equation of [6]) seem to have mass limits beyond which the wave functions become pathological. These Hamiltonians are all so complicated that it seems impossible to find analytic solutions to assist in the interpretation. In [5], Corichi et al. have given a series of numerical solutions that give the evolution of wave functions that are initially sharply peaked over a radius near the classical horizon, $R = 2M$, for the $\cosh P$ Hamiltonian (2) and the Hamiltonian given by Eqs. (3-4). All of these solutions show evolution of the peak toward $R = 0$ with a bounce caused by the boundary conditions at $R = 0$ with the appearance of interference fringes as well as a rapid spread of the wave packet.

The Kuchař-Hájíček Hamiltonian has wave functions similar to those of (2), but with many rapid oscillations superimposed.

Since these Hamiltonians are self-adjoint, unitary evolution implies that a peak formed from scattering states will always rebound to $R = \infty$. One can ask whether this behavior means that all quantum collapse of this sort implies a rebound into our own universe. Since τ is proper time on the shell and R is also a shell variable, such questions can only be answered by knowing the global quantum spacetime surrounding the shell. In any case, the scenario of the shell observer is that he sees (begging questions of quantum measurement and the reduction of the wave packet) the shell collapse to some point near $R = 0$, where uncertainty principle effects change the classical equations of motion and the shell rebounds (actually, a shell where the particles do not interact directly with one another "passes through itself" and reexpands, that is, each radially infalling particle passes through $R = 0$ and the azimuthal angle θ jumps from the initial θ_0 to $\theta_0 + \pi$). Even if the shell has collapsed below its classical horizon, in finite proper time it will again be above the horizon and traveling toward $R = \infty$. This quasi-classical scenario is not surprising. In proper time, a classical shell that manages to avoid forming a curvature singularity at $R = 0$ would behave in this way, but as R becomes greater than $2M$ the shell would be in a universe past our temporal infinity (i_+), or in "another universe."

This quasi-classical scenario is what one might expect to see for a large mass where the quantum fluctuations would be small compared to the mass and the evolution of the wave packet would be coherent long enough for the shell to collapse past its horizon and the shell would emerge from the horizon into a new universe. However, if the wave packet spreads sufficiently so that the width is greater than the classical horizon radius, we can see that a horizon might never form, and the shell would reexpand into our own universe.

The problem of horizon formation in the quantum system is very difficult. Event horizons are global features and one has to try to define a global feature in a fluctuating manifold. Of course, this quantum manifold must be constructed in terms of the full minisuperspace canonical quantum gravity of the shell-metric system. In the shell case we tend to use some kind of approximation to construct the spacetime metric. Kuchař [11] argues that for the simple shell minisuperspace we may just replace the the shell mass (Schwarzschild mass) and the shell radius in the metric outside the shell by the corresponding operators to make a "metric operator." The problem with this metric operator is that it is a function of shell proper time, and studies of the metric close to the shell [12] cannot tell us whether a true event horizon (tied to observer time at infinity) forms.

Other approximations are under study [9]. The simplest calculation is to calculate $< \hat{R}(\tau) >$ and the uncertainty $\Delta R = \sqrt{< (\hat{R} - < \hat{R} >)^2 >}$ and check whether ΔR becomes very large as $< \hat{R} >$ becomes small so that $< \hat{R} >$ does not fall below the classical horizon and ΔR is larger than the classical horizon, which can be taken as an indication of the non-formation of a horizon. In [9], numerical evaluations of these two quantities will be presented for the Hamiltonians (2) and (3-4), and they suggest that no horizon forms for small masses. Another possibility (to be considered in [9]) would be to take $\mathrm{M}|\psi(R,\tau)|^2$ to be a classical density $\rho(R,\tau)$ and calculate the classical metric due to a classical fluid with this mass distribution and see whether a horizon forms. Note that M should be either the rest mass or the Schwarzschild mass. It is not yet clear which. There are technical problems with this calculation. We have to calculate the metric from a density that is given in terms of a solution of the Schrödinger equation for our Hamiltonian, and there is no guarantee that this density can be made to obey the equation $T^{\mu\nu}{}_{;\nu} = 0$ for our fluid. Another possibility considered in [9] is that of the "metric operator" mentioned above, which was extended to the whole manifold outside the shell and used to define a "quantum" stress-energy tensor.

The rotationless 2 + 1 problem has some advantages over the 3 + 1 problem. The Hamiltonian can be constructed fairly easily, and, as will be shown, has the form of a harmonic oscillator. The Schrödinger equation for this Hamiltonian has well-known analytic solutions. The expectation value of R and its uncertainty can, in principle, be calculated analytically. The analogue of the other calculation using $\rho(R,\tau)$ is much simpler than in the 3 + 1 case. The classical horizon is easily found. In the next section these ideas will be considered.

3 The classical problem of shells in 2 + 1 gravity

The first element we need for this problem is an equation for the radius of the shell. This problem has been studied in detail by Peleg and Steif [1], using the 2 + 1 version of the original formulation of Israel [2], and Crisóstomo and Olea [3], using canonical methods. We will review the calculation of Peleg and Steif (using the parametrization of the metric of Crisóstomo and Olea).

The calculation we will use follows that of Israel who, as mentioned above, studied the collapse of a shell in 3 + 1 gravity, represented by a delta-function sphere of dust of radius $R(\tau)$. As we have mentioned, in 2 + 1 gravity the shell is a circle, i.e. a ring of matter, also of radius $R(\tau)$. The metric of spacetime will be written in circular coordinates, where flat space is represented by the metric

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2. \tag{5}$$

The equation for a circle $R(\tau)$ is

$${}^{(3)}r = R(\tau), \qquad {}^{(3)}\theta = \theta, \qquad {}^{(3)}t = t(\tau). \tag{6}$$

We will use the notation $i, j = 1, 2, 3$ and $A, B = 1, 2$. We will now need a set of coordinates ξ_A on the circle, which we will take to be $\xi_A = (\tau, \theta)$, and we will define two three-vectors on the circle by $e^i_{(A)} \equiv \partial x^i/\partial \xi_A$, where x^i are the coordinates t, r, θ given above. We have

$$e^i_{(\tau)} = (\frac{dt}{d\tau}, \dot{R}, 0), \tag{7}$$

$$e^i_{(\theta)} = (0, 0, 1). \tag{8}$$

The metric on the circle is ($ds^2 = g_{AB} d\xi^A d\xi^B$)

$$ds^2 = -d\tau^2 + R^2(\tau)d\theta^2. \tag{9}$$

We now have to calculate the three-dimensional metric due to the ring of matter. We have to solve the 2 + 1 Einstein equations (necessarily with cosmological constant Λ to avoid completely flat solutions),

$$R_{ij} - \frac{1}{2} g_{ij} R = -\Lambda g_{ij}, \tag{10}$$

inside and outside the circle. Since there is a Birkhoff theorem in 2 + 1 gravity, these metrics in vacuum will be static or stationary. We will study the static case, where the matter has no angular momentum. In this case the metric has the form

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\theta^2, \tag{11}$$

and the well-known solution is

$$f = B - \Lambda r^2, \tag{12}$$

B a constant. Since Λ has units of inverse length, we will write, as is common, $\Lambda = \pm 1/\ell^2$. The final form of the static, circularly-symmetric metric is

$$ds^2 = -\left(1 - 2M \mp \frac{r^2}{\ell^2}\right)dt^2 + \frac{1}{\left(1 - 2M \mp \frac{r^2}{\ell^2}\right)}dr^2 + r^2 d\theta^2, \tag{13}$$

where we have taken B, following Crióstomo and Olea, to be $1 - 2M$, M a Schwarzschild mass. Notice that this is not the choice of Bañados, Teitelboim and Zanelli (BTZ) [13], who take $B = -M_{\mathrm{BTZ}}$, but we choose $1 - 2M$ following Cristó somo and Olea because we will want the metric inside the ring to be the 2 + 1 AdS metric. These two choices of B correspond to a shift in the zero of the mass parameter [14], [15]. The relation between these two masses (and others) and the ADM mass is discussed in the Appendix. This choice requires that the mass term be negative in order to have a 2 + 1 black hole with a horizon. Notice also that we must have $M > 1/2$ in order to have a horizon. As in the 3 + 1 case, we expect that outside the ring we will have a black hole metric with some "Schwarzschild mass" M, that is,

$$ds^2 = -\left(1 - 2M + \frac{r^2}{\ell^2}\right)dt^2 + \frac{1}{\left(1 - 2M + \frac{r^2}{\ell^2}\right)}dr^2 + r^2 d\theta^2, \tag{14}$$

and inside the ring we will have a metric that is as near as possible to flat space (but with a cosmological constant). This will be (14) with $M = 0$, that is the 2 + 1 AdS metric,

$$ds^2 = -\left(1 + \frac{r^2}{\ell^2}\right)dt^2 + \frac{1}{\left(1 + \frac{r^2}{\ell^2}\right)}dr^2 + r^2 d\theta^2. \tag{15}$$

With these preliminaries we can follow the steps of Israel's derivation of the equation of motion of a shell in 3 + 1 gravity, a manner similar to that used by Peleg and Steif [1]. For any three-dimensional vector A_A, we can define components of on the surface of the circle as

$$A_A \equiv A_i e^i_{(A)}, \qquad A^i \equiv A^A e^i_{(A)}. \tag{16}$$

We will also need the vector n^i, the unit normal to the circle $R(\tau)$, with $n^i n_i = +1$. We can now use the standard equation

$$\frac{\partial A^i}{\partial \xi^A} = A^B{}_{;A} e^i_{(B)} - A^B K_{BA} n^i, \tag{17}$$

where K_{AB} is the extrinsic curvature of the ring, to arrive at the equation

$$\frac{\partial u^i}{\partial s}\Big|^{\pm} = \frac{\partial u^i}{\partial \xi^A}\frac{d\xi^A}{ds} = -U^A K^{\pm}_{AB} u^B n^i, \tag{18}$$

or

$$n_i u^i{}_{;j} u^j = -u^A u^B K_{AB}\Big|^{\pm}, \tag{19}$$

where $\pm$ means the quantity calculated for $r > R(\tau)$ (+) and $r < R(\tau)$ (-). In order to calculate the left-hand-side of (19) we will need expressions for u^i and n_i. Following Israel, $u^r = \dot{R}$, $u^\theta = 0$ and $u_i u^i = -1$ gives us $u^t \equiv X = \sqrt{f + \dot{R}^2}/f$. There are many ways to find n_i, and we obtain $n_t = -\dot{R}$, $n_r = X$ and $n_\theta = 0$. We will also need the definition,

$$\gamma_{AB} \equiv K^+_{AB} - K^-_{AB}, \tag{20}$$

and the Lanczos relation,

$$\gamma_{AB} - g_{AB}\gamma = 8\pi S_{AB}, \tag{21}$$

where g_{AB} are the components of the induced metric on the surface in terms of τ and θ, $\gamma = g^{AB}\gamma_{AB}$ and S_{AB} is the surface stress-energy tensor. In our case, we will be interested in a dust shell, so we will take $S_{AB} = \sigma u_A u_B$, σ the rest mass density on the ring.

Following Crisóstomo and Olea, the Einstein equation for $G_{ij}n^i n^j$ becomes

$${}^2R + K^{\pm}_{AB}K^{AB}_{\pm} - K^2_{\pm} = -2G_{ij}n^i n^j = -\Lambda g_{ij} n^i n^j = -\Lambda. \tag{22}$$

Subtracting the minus equation from the plus equation, and defining $\tilde{K}_{AB} \equiv \frac{1}{2}(K^+_{AB} + K^-_{AB})$, we arrive at

$$\gamma_{AB}\tilde{K}^{AB} - \tilde{K}\gamma = 0, \tag{23}$$

where $K = g_{AB}K^{AB}$. The Lanczos relation gives us $S_{AB}\tilde{K}^{AB} = 0$ or, since $S_{AB} = \sigma u^A u^B$,

$$u^A u^B \tilde{K}_{AB} = 0. \tag{24}$$

Israel uses Eq.(19), calculating the left-hand-side directly, and, doing the same for our case, we find

$$(n_i u^i{}_{;j} u^j)^{\pm} = \frac{1}{f^{\pm} X^{\pm}} \left(\ddot{R} + \frac{R}{\ell^2} \right). \tag{25}$$

Now, summing the plus and minus versions, and using $u^A u^B \tilde{K}_{AB} = 0$, we find

$$\left(\ddot{R} + \frac{R}{\ell^2} \right) \left[\frac{1}{\sqrt{f^+ + \dot{R}^2}} + \frac{1}{\sqrt{f^- + \dot{R}^2}} \right] = 0. \tag{26}$$

Using $\gamma_{AB} = 8\pi(S_{AB} - g_{AB} S)$ and calculating $\gamma_{AB} u^A u^B = u^A u^B (S_{AB} - g_{AB} S = \sigma - \sigma = 0$, we find that

$$\left(\ddot{R} + \frac{R}{\ell^2} \right) \left[\frac{1}{\sqrt{f^+ + \dot{R}^2}} - \frac{1}{\sqrt{f^- + \dot{R}^2}} \right] = 0, \tag{27}$$

which is consistent. Unfortunately, Israel uses the 3 + 1 equivalent of (27), where the right-hand-side is *not* zero, to find a second-order equation of motion for $R(\tau)$ in terms of the rest mass density, but our equations no longer contain any reference to this variable. However, we can use, following Peleg and Steif [1], the Lanczos relation, Eq.(21), directly. If we impose the condition $\ddot{R} + R/\ell^2 = 0$, we have that $(n_i u^i{}_{;j} u^j)^{\pm}$ are both zero, so from Eq. (19), $u^A u^B K^{\pm}_{AB} = 0$. From $u_A \equiv u_i e^i_{(A)}$, we find that $u^A = (1, 0)$, so $K^{\pm}_{\tau\tau} = 0$. Now, from the Lanczos relation we have that $\gamma_{\tau\tau} - g_{\tau\tau}\gamma = 8\pi S_{\tau\tau}$, but $\gamma_{\tau\tau} = 0$ and $g_{\tau\tau} = -1$, so $S_{\tau\tau} = \sigma$, and finally,

$$\gamma = 8\pi\sigma. \tag{28}$$

Since $\gamma = g^{AB}\gamma_{AB} = -\gamma_{\tau\tau} + (1/R^2)\gamma_{\theta\theta} = (1/R^2)\gamma_{\theta\theta}$, we have

$$\gamma_{\theta\theta} = 8\pi R^2 \sigma. \tag{29}$$

We can now calculate $K_{\theta\theta}$ directly from its definition, $K_{\theta\theta} = n_{\theta,\theta} -^{(3)} \Gamma^k_{\theta\theta}\big|_{r=R}$. Calculating the necessary Γ^i_{jk} and using the fact that $n_{\theta,\theta} = 0$, we find that $K_{\theta\theta} = -fXR$, so

$$\gamma_{\theta\theta} = (fX)^- R - (fX)^+ R. \tag{30}$$

From $S^{AB}_{\ ;B} = 0$, we can show that $\sigma = m/2\pi R$, m the total rest mass of the ring.

If we now consider the equation of motion of R, $\ddot{R} = -R/\ell^2$, we see that it has a first integral,

$$E = \frac{1}{2}\dot{R}^2 + \frac{R^2}{2\ell^2}. \tag{31}$$

Using $(fX)^- R - (fX)^+ R = 4mR$, and inserting (31) in this relation, we find

$$\sqrt{1 + 2E} - \sqrt{1 - 2M + 2E} = 4m, \tag{32}$$

which can be solved for E as

$$E = \frac{M^2}{2(4m)^2} + \frac{M}{2} + 2m^2 - \frac{1}{2} = \frac{1}{32}\left(\frac{M}{m} + 8m\right)^2 - \frac{1}{2}. \tag{33}$$

We would like to use Equation (31) to construct a Hamiltonian formulation of the problem. This equation is simply the energy equation for a harmonic oscillator, so it is obvious that if we take E as our Hamiltonian we can define a momentum P_R as $\dot{R}$, and we have

$$H = \frac{1}{2}P_R^2 + \frac{R^2}{2\ell^2}. \tag{34}$$

Hamilton's equations for this Hamiltonian are equivalent to the equation of motion $\ddot{R} = -R/\ell^2$.

4 Quantum collapse in 2 + 1 gravity

4.1 Schrödinger equation and evolution of wave packets

Once we have a Hamiltonian in terms of the ring variables R and τ, we can try to construct a Schrödinger equation for the problem. Before we do this, we have to study the units of our variables. If we write the classical relation

$$\left(\frac{M}{4m} + 2m\right)^2 - 1 = \dot{R}^2 + \frac{R^2}{\ell^2}, \tag{35}$$

we see that the quantities on the right-hand-side have no units, so the left-hand-side must also be dimensionless, so M and m must be dimensionless. We would like to multiply the true mass by some combination of G and c that will give a dimensionless mass. We have to be careful about the law of gravitation in two space dimensions. In general, Gauss's law would give us, in d dimensions, $F = GMm/r^{d-1}$, and in two space dimensions $F = GMm/r$. This means that G would have units of $[M]^{-1}[L]^2[T]^{-2}$, and G/c^2 has units of $1/[M]$, and GM/c^2 is dimensionless. If we consider the metric function $f(r) = 1-2M+R^2/\ell^2$, which has to be dimensionless, this means that M in conventional units is GM/c^2. It is usual (see Ref. [15]) to define a "Planck mass" $M_{\mathrm{Pl}} = c^2/G$ and a "Planck length" $L_{\mathrm{Pl}} = \hbar G/c^3$, even though there is no particular reason to identify this mass and length with the 3 + 1 Planck mass and Planck length, especially if we have no knowledge of what G should be in two space dimensions.

In our case, M means M/M_{Pl} and m means m/M_{Pl}. In conventional units, our equation for R becomes

$$\left(\frac{M}{4m} + \frac{2m}{M_{\mathrm{Pl}}}\right)^2 - 1 = \frac{1}{c^2}\left(\frac{dR}{d\tau}\right)^2 + \frac{R^2}{\ell^2}. \tag{36}$$

Our "Hamiltonian" should have the units of energy for a conventional quantum Hamiltonian, so we should multiply Eq. (35) above by $M_{\mathrm{Pl}}c^2$, our R-momentum

P_R becomes $M_{\rm Pl}\dot{R}$, and with

$$H = \frac{M_{\rm Pl}c^2}{2}\left(\frac{M}{4m}\right)^2 + \frac{Mc^2}{2} + 2mc^2\left(\frac{m}{M_{\rm Pl}}\right)^2 - \frac{M_{\rm Pl}c^2}{2}, \tag{37}$$

and

$$H = \frac{P_R^2}{2M_{\rm Pl}} + \frac{M_{\rm Pl}}{2}\omega_0^2 R^2, \tag{38}$$

with $\omega_0 = c/\ell$.

We want to quantize this system with H replaced by an operator $\hat{H}$. If we look at (37), we see that the right-hand-side must become an operator, so the left-hand-side must also be an operator. One way to achieve this is to take M to be a q-number, and m and $M_{\rm Pl}$ c-numbers. Of course, there is no special reason to make this choice, but we do so to make contact with previous work (see, for example, [11]). If we consider Eq. (2), this choice is motivated by the fact that the Schrödinger equation for (2) is similar to the hydrogen atom Schrödinger equation, with m playing the role of e^2, and it is usual in the hydrogen atom to take e^2 as a c-number rather than a q-number.

We can now take the wave function of the system to be $\tilde{\Psi} = \Psi_M\Psi(R,\tau)$, with $\Psi_M(M)$ an approximate eigenstate of $\hat{M}$ with eigenvalue M_0. An exact eigenstate of $\hat{M}$ of this type would be $\delta(M - M_0)$, but to avoid problems with the integral of the square of a delta function we will assume that Ψ_M is an extremely sharply peaked wave function centered on $M = M_0$. In this case, our Schrödinger equation becomes ($\hat{M}^2\Psi_M \approx M_0^2\Psi_M$, and realizing $\hat{P}_R$ as $-i\hbar\partial/\partial R$)

$$\begin{aligned}&\left[\frac{M_{\rm Pl}c^2}{2}\left(\frac{M_0}{4m}\right)^2 + \frac{M_0c^2}{2} + 2mc^2\left(\frac{m}{M_{\rm Pl}}\right)^2 - \frac{M_{\rm Pl}c^2}{2}\right] = \\ &= i\hbar\frac{\partial\Psi(R,\tau)}{\partial\tau} = -\frac{\hbar^2}{2M_{\rm Pl}}\frac{\partial^2\Psi(R,\tau)}{\partial R^2} + \frac{M_{\rm Pl}}{2}\omega_0^2R^2\Psi(R,\tau).\end{aligned} \tag{39}$$

One curious fact about this equation is that, if we define $\Psi = \exp(-iE\tau/\hbar)\psi(R)$, then the energy eigenvalues are

$$E_n = (n + \frac{1}{2})\hbar\omega_0 = (n + \frac{1}{2})\frac{\hbar c}{\ell}, \tag{40}$$

and since E is given by the first line of (39), there is a discrete relation between the Schwarzschild mass, M_0, and the rest mass, m,

$$\left(\frac{M_0}{4m} + \frac{2m}{M_{\rm Pl}}\right)^2 - 1 = (2n+1)\frac{\hbar G}{c^3\ell} = (2n+1)\frac{L_{\rm Pl}}{\ell}. \tag{41}$$

We will now return to units where $G = c = \hbar = 1$, M_0 now meaning $M_0/M_{\rm Pl}$, m now meaning $m/M_{\rm Pl}$, and ℓ meaning $\ell/L_{\rm Pl}$. If we solve for M_0 in terms of m and ℓ, we find

$$M_0 = 8m^2\left[\sqrt{\frac{1}{4m^2} + \frac{1}{2m^2}\left[\frac{1}{\ell}\left(n + \frac{1}{2}\right)\right]} - 1\right]. \tag{42}$$

We now want to write our Schrödinger equation in terms of dimensionless variables. For a moment we will return to conventional units, and define the following dimensionless variables. We define $y = R/\sqrt{\ell L_{\rm Pl}}$ and $T = c\tau/\ell$. The Schrödinger equation now becomes

$$i\frac{\partial\Psi(y,T)}{\partial T} = -\frac{\partial^2\Psi(y,T)}{\partial y^2} + y^2\Psi(y,T) \tag{43}$$

which has eigensolutions

$$\Psi = e^{-i(n+\frac{1}{2})T}\frac{1}{\sqrt{2^n n!}(\pi)^{1/4}}e^{-y^2/2}H_n(y), \tag{44}$$

H_n Hermite polynomials.

Now, in order to make contact with previous work, we would like to study the evolution of a wave packet sharply peaked around a value of $y = y_0$, some distance outside the point where a classical horizon would form if the radius of the shell were to fall below $R_H = \ell\sqrt{2M_0 - 1}$ and follow its movement as the packet falls toward $R = 0$. In previous work it was necessary to solve this problem numerically, since in the $3+1$ case analytic solutions to the Schrödinger equation did not exist [5] [9], and in the Newtonian case, [8], while analytic solutions existed, it was impossible to sum the expansions for the relevant wave functions to give an analytic expression for the collapsing wave packet. In the $2+1$ case, however, we can give an exact analytic expression for the wave packet as a coherent harmonic-oscillator state. Even though the eigensolutions are nothing but harmonic oscillator wave functions, the radial variable R cannot be negative. In order to keep this from happening we will take the potential to be that of a half oscillator with an infinitely hard wall at $R = 0$. This potential is shown in Figure 1. This will mean that $\Psi(0,\tau)$ will always be zero. This boundary condition can be enforced by only expanding in odd-n harmonic oscillator eigenfunctions. We will want to begin with a difference of two Gaussian states, one peaked around $y = +y_0$ and the other around $y = -y_0$, so that their sum at $y = 0$ is zero. This state is

$$\Psi(y,0) = \alpha[e^{-\frac{1}{2}(y-y_0)^2} - e^{-\frac{1}{2}(y+y_0)^2}], \tag{45}$$

only valid for $y > 0$, with α a normalization constant. Since this is a sum of two Gaussian states, we can use standard techniques to construct the difference between two coherent Gaussian states with (45) as an initial condition. The result is

$$\begin{aligned}\Psi(y,\tau) = \alpha e^{-i\omega_0\tau/2}e^{iy_0^2\sin 2\omega_0\tau/4}[&\exp(-\frac{1}{2}\{(y-y_0\cos\omega_0\tau)^2\})\exp(-iyy_0\sin\omega_0\tau) - \\ &- \exp(-\frac{1}{2}\{(y+y_0\cos\omega_0\tau)^2\})\exp(iyy_0\sin\omega_0\tau)],\end{aligned} \tag{46}$$

which is zero at $y = 0$ for all τ. The normalization α is easily found to be $\alpha^2 = [\sqrt{\ell}\sqrt{\pi}(1 - e^{-y_0^2})]^{-1}$.

We would like to connect the variables that describe the wave function with the radius of the classical horizon, R_H, shown in Fig. 1. We will make the following definitions, $y_0 = \lambda\sqrt{\ell}\sqrt{2M_0 - 1}$, $y = w\sqrt{\ell}\sqrt{2M_0 - 1}$, $T = c\tau/\ell$. If we take $\sqrt{\ell} = 1/\sqrt{2M_0 - 1}$, the unnormalized probability density, $\rho \equiv \Psi^*\Psi/\alpha^2$,

$$\rho = e^{-(w-\lambda\cos T)^2} + e^{(w+\lambda\cos T)^2} - \\ -2e^{-w^2}e^{-\lambda^2\cos^2 T}\cos(2\lambda w\sin T), \tag{47}$$

is shown, for $\lambda = 3$, as a function of w for various values of T in Figures 2-7. We begin with a Gaussian packet at $T = 0$ that collapses toward $w = 0$, developing interference fringes as it reaches a minimum for some $w < \lambda$, then rebounds toward $w = \lambda$ again. This pattern then repeats forever. In order to compare our results with earlier work, we will only be interested in one cycle of this pattern.

4.2 The formation of a horizon

In previous articles the quantization of the shell collapse was used to study the possibility of the formation of a horizon in the quantum collapse. As mentioned in Sec. 2, this concept has many difficulties. An event horizon is a global construct and it has no local definition. This means that in quantum gravity one would have to return to the starting point and try to define what a "quantum horizon" might be. Once this definition has been decided upon, one must try to find out if some collapse process will result in the formation of such a horizon, with the result being a probability of horizon formation. Theories of quantum gravity in their present state are far from being able to give us this result, so in shell collapse some articles [5], [9], [8] have tried to give an estimate of horizon formation by finding out if a sharply peaked wave packet, during its collapse toward $R = 0$, will fall, in some sense, below the classical horizon radius, R_H. In some sense, because the packet will usually spread and basically will never lie entirely below $R = R_H$. One could use the integral of $\Psi^*\Psi$ from $R = 0$ to $R = R_H$, which is a number less than one, as the probability of horizon formation. Another possibility would be to use the operator $\hat{M}$ in the expression $\hat{R}_H = \ell\sqrt{2\hat{M} - 1}$ to define a "horizon operator" that could be used to define the probability of horizon formation. In previous work it was decided to use $< \hat{R} >$ as a function of τ, which will fall from R_0 to a minimum and then begin to increase again. If this minimum is below the classical horizon, one can say that a horizon forms and if not, not. This is a yes-no answer instead of a probability, but it is a quick estimate. We will use this concept below.

It is not difficult to calculate $< \hat{R} > (\tau)$ from the wave packet given in (46), but the result is a complicated function that contains the error function and Dawson's function, so trying to find the minimum of $< \hat{R} > (\tau)$ by finding the point where $d < \hat{R} > /d\tau = 0$ requires the solution of a transcendental algebraic equation, making it difficult to give an analytic expression for the point where a horizon would form. Instead we will use the fact that, as can be seen from Fig. 5, that at $T = \pi/2$, the point where the packet begins to rebound, the peak nearest $R = 0$ is high and narrow. We can use the position of this peak as a parameter to tell us whether a horizon forms or not. If the position of the peak is below R_H, a horizon forms, and if not, not. In previous work it was found that for large masses the shell collapse was so rapid that the wave packet fell below R_H so quickly that quantum mechanics did not allow it to rebound before that point, while for small masses the rebound occurred for $R > R_H$. In order to calculate the position of the maximum of ρ nearest to $w = 0$, we write ρ as

$$\rho(y,T) = 2e^{-y_0^2 - y_0^2 \cos^2 T}[\cosh(2yy_0 \cos T) - \cos(2yy_0 \sin T)]. \tag{48}$$

The packet is closest to $y = 0$ when $T = \pi/2$, which gives

$$\rho(y, \pi/2) = 2e^{-y^2}[1 - \cos(2yy_0)] = 4e^{-y^2}\sin^2(yy_0). \tag{49}$$

This function has zeros at $y = n\pi/y_0$, and either peaks or valleys (or inflection points) at $d\rho(y, \pi/2)/dy = 0$, or

$$8e^{-y^2}\sin(yy_0)[-y\sin(yy_0) + y_0\cos(yy_0)] = 0. \tag{50}$$

An obvious solution to this equation is $y = n\pi/y_0$ for the zeros. Another possibilty is $y = y_c$, where

$$y_c = y_0 \cot(y_c y_0), \tag{51}$$

which are peaks (it is easy to show that $d^2\rho/dy^2 < 0$). If we define $y_c = \gamma\sqrt{\ell}\sqrt{2M_0 - 1}$, we have

$$\gamma = \lambda \cot[\gamma\lambda\ell(2M_0 - 1)]. \tag{52}$$

for $\lambda = 3$ we can solve for M_0 as

$$M_0 = \frac{1}{2}\left[\frac{1}{3\gamma\ell}\cot^{-1}\left(\frac{\gamma}{3}\right) + 1\right]. \tag{53}$$

We now have two relations, (53) and $(M_0/4m + 2m)^2 - 1 = E$. The minimum for M_0 [the right-hand-side of (53) decreases monotonically with γ] is when $\gamma = 3$, since we want $y_c \leq y_0$. The numerical value of this minimum depends on ℓ, and there is no simple reason to choose any special value for ℓ. We can use the relation between M_0, m and E to find another value for M_0 as a function of ℓ, and equating these two, we get a numerical value for ℓ. There are various ways to define M_0 as a function of ℓ from the energy relation. If we consider the classical equation

$$\left(\frac{M_0}{4m} + 2m\right)^2 - 1 = \ell p_y^2 + \frac{y^2}{\ell}, \tag{54}$$

and take y_0 to be where $p_y = 0$, then $(M_0/4m + 2m)^2 - 1 = y_0^2\ell = 18M_0 - 9$. One solution of this quadratic equation is

$$M_0 = 8m^2\left[17 + \sqrt{288 - \frac{2}{m^2}}\right]. \tag{55}$$

This is real for $18m^2 > 1/18$, which gives a minimum for M_0 equal to $M_0 = 17/18$, and for $\gamma = 3$,

$$M_0 = \frac{1}{2}\left(\frac{\pi}{36\ell} + 1\right) = \frac{17}{18}, \tag{56}$$

or $\ell = \pi/32 \approx 0.098$, and $M_0(\gamma = 3) = 17/18 \approx 0.944$.

If we use the expectation value of $\hat{H}$ as our energy, we have $< \hat{H} >= (M_0/4m + 2m)^2 - 1$, and calculate $< \hat{H} >$ to be

$$< \hat{H} >= \frac{1}{2}[1 + y_0^2(1 + e^{-y_0^2})(1 - e^{-y_0^2})^{-1}], \tag{57}$$

in principle we can solve (using $y_0 = 9\ell[2M_0 - 1]$) for M_0, but the transcendental equation is difficult to solve for M_0 as a function of m and ℓ. If we were able to assume that y_0 were sufficiently large that the the factor $(1+e^{-y_0^2})(1-e^{-y_0^2})^{-1}$ were one, we could solve the resulting quadratic for M_0 easily. However, for the minimum M_0 from (53) with $\gamma = 3$, we find that $y_0^2 = \pi/4$, and our factor is almost equal to three. We can find an approximate solution by inserting $y_0^2 = \pi/4$ in the factor and write the quadratic equation

$$\left(\frac{M_0}{4m} + 2m\right) - 1 = \frac{1}{\ell} + (18M_0 - 9)(1 + e^{-\pi/4})(1 - e^{-\pi/4})^{-1}, \tag{58}$$

which gives, using $\beta \equiv (1 + e^{-\pi/4})(1 - e^{-\pi/4})^{-1} = \coth(\pi/8) \approx 2.68$

$$M_0 = 8m^2\left(18\beta - 1 + \sqrt{(18\beta)^2 - 36\beta - \frac{1}{4m^2}\left[(9\beta - 1) - \frac{1}{\ell}\right]}\right), \tag{59}$$

which has real solutions for

$$4m^2 > \frac{(9\beta - 1) - 1/\ell}{(18\beta)^2 - 36\beta}, \tag{60}$$

which has a minimum at $4m^2 = [(9\beta - 1) - 1/\ell]/[(18\beta)^2 - 36\beta]$, and inserting this expression in (59), and equating the result to (53) with $\gamma = 3$, we find $\ell(1/9\beta - 1)[\pi\beta/4 + (18\beta - 1)/(9\beta - 1)] \approx 0.180$. This gives us $M_0(\gamma = 3) = 0.742$.

For the classical approximation we now have $M_0(\gamma = 1) = 2.62$, while for $< \hat{H} >$ we have $M_0(\gamma = 1) = 1.66$. Since $< \hat{H} >$ should give an answer closer to the correct one, we can take it as giving the best value of ℓ. This means that for

$$0.742 < M_0 < 1.66, \tag{61}$$

no horizon forms, while for $M_0 > 1.66$ one does. For example, for $\gamma = 1/2$, $M_0 = 2.80$.

From the Appendix, we can give these masses in terms of M_{BTZ}, and for

$$1.48 < M_{\mathrm{BTZ}} < 3.32, \tag{62}$$

no horizon forms, while for $M_{\mathrm{BTZ}} > 3.32$ one does.

Of course, as has been mentioned above, the rebound of the wave packet, once it has passed the horizon does not mean that the shell returns through the horizon into the same spacetime where it began. In spite of the fact that in terms of proper time the shell exits the horizon in a finite time, in terms of the time, t, of an observer at infinity this exit occurs at time *after* $t = \infty$, or into "another universe."

5 Conclusions

The 2 + 1 calculation given above has the advantage of being the only model so far that has given analytic solutions to the quantum collapse problem. This is a great advantage in understanding the origin of features in the numerical solutions given before in Refs. [5], [9], [8], especially the the existence of the interference fringes that were noticed in Refs. [5] and [9]. The results for horizon formation are at least consistent with those of previous work.

Another possible model problem would be to study a simple relativistic model of gravity such as the Nordström theory. In a such a scalar theory the reduction to a Hamiltonian formulation is relatively straightforward, but one of us (L. O.) (see Ref. [16]) has shown that the Hamiltonian is more complicated than that of ordinary 3 + 1 gravity. He has also shown that linearized gravity gives the same Hamiltonian as full gravity.

Still another possibility would be to carry out the program suggested in Ref. [9] in the 2 + 1 case, that is, to try to define an average metric by writing the metric outside the ring in terms of coordinates ξ (where $r = R(\tau) + \xi$) and $t = \tau$. In this metric we can replace $R(\tau)$ by $\hat{R}$, to form a metric operator $\hat{g}_{ij}$ and calculate $< \hat{g}_{ij} > (\xi, \tau)$, where, for example, $\hat{g}_{\theta\theta} = (\hat{R} + \xi)^2$, so

$$< \hat{g}_{\theta\theta} >=< \hat{R}^2 > (\tau) + 2 < \hat{R}(\tau) > \xi + \xi^2, \tag{63}$$

and use the classical Einstein tensor, G_{ij}, calculated from $< \hat{g}_{ij} >$ as $8\pi T_{ij}$ to define a classical stress-energy tensor $T_{ij}(\xi, \tau)$ (which automatically satisfies $T^{ij}{}_{;j} = 0$). From this T_{ij}, we can define, for example, a shell density as a function of ξ and τ, which could be plotted.

Appendix

The mass parameter M that appears in the black hole metric (14) can be associated with the ADM mass of the black hole. The ADM mass for the metric with $B = -M$ is defined as [14], [15]

$$M_{\rm ADM} = \frac{M_{\rm Pl}}{16\pi} \int dr\, d\theta \sqrt{^{(2)}g}\, ^{(2)}R \approx \frac{\ell}{r} \frac{M_{\rm Pl}}{8} M_{\rm BTZ}, \tag{64}$$

where $^{(2)}g_{AB}$ is the two-metric on surfaces of constant t, and $^{(2)}R$ is the Ricci scalar on these surfaces. The factor ℓ/r is related to proper time at $r \to \infty$ [15]

and is usually ignored, so we find that

$$M_{\rm BTZ} = \frac{8}{M_{\rm Pl}} M_{\rm ADM}. \tag{65}$$

We can choose our mass unit to be any multiple of $M_{\rm Pl}$. In [14] and [15] they take it to be $8M_{\rm Pl}$ while in [1] it is taken to be simply $M_{\rm Pl}$.

If we calculate $M_{\rm ADM}$ for our metric, we find that

$$M_{\rm ADM} = \frac{4}{M_{\rm Pl}} \frac{\ell}{r}(M - \frac{1}{2}) + A, \tag{66}$$

where A is a constant taken to be $(M_{\rm Pl}/4)r/\ell$ in [14] and [15] that represents the zero of the ADM energy, $M_{\rm ADM}c^2$. In older work [13], BTZ chose $A = (M_{\rm Pl}/4)r/\ell + (M_{\rm Pl}/8)\ell/r$ as we have in this article. The difference lies in the metric for $M = 0$. In our case, $M = 0$ is AdS, while for later BTZ work $M = 0$ represents

$$ds^2 = -\frac{r^2}{\ell^2}dt^2 + \frac{\ell^2}{r^2}dr^2 + r^2 d\theta^2. \tag{67}$$

In our case, the choice of reference mass equal to $M_{\rm Pl}$ leads to a relation between our M and $M_{\rm BTZ}$,

$$M = \frac{1}{2} M_{\rm BTZ}. \tag{68}$$

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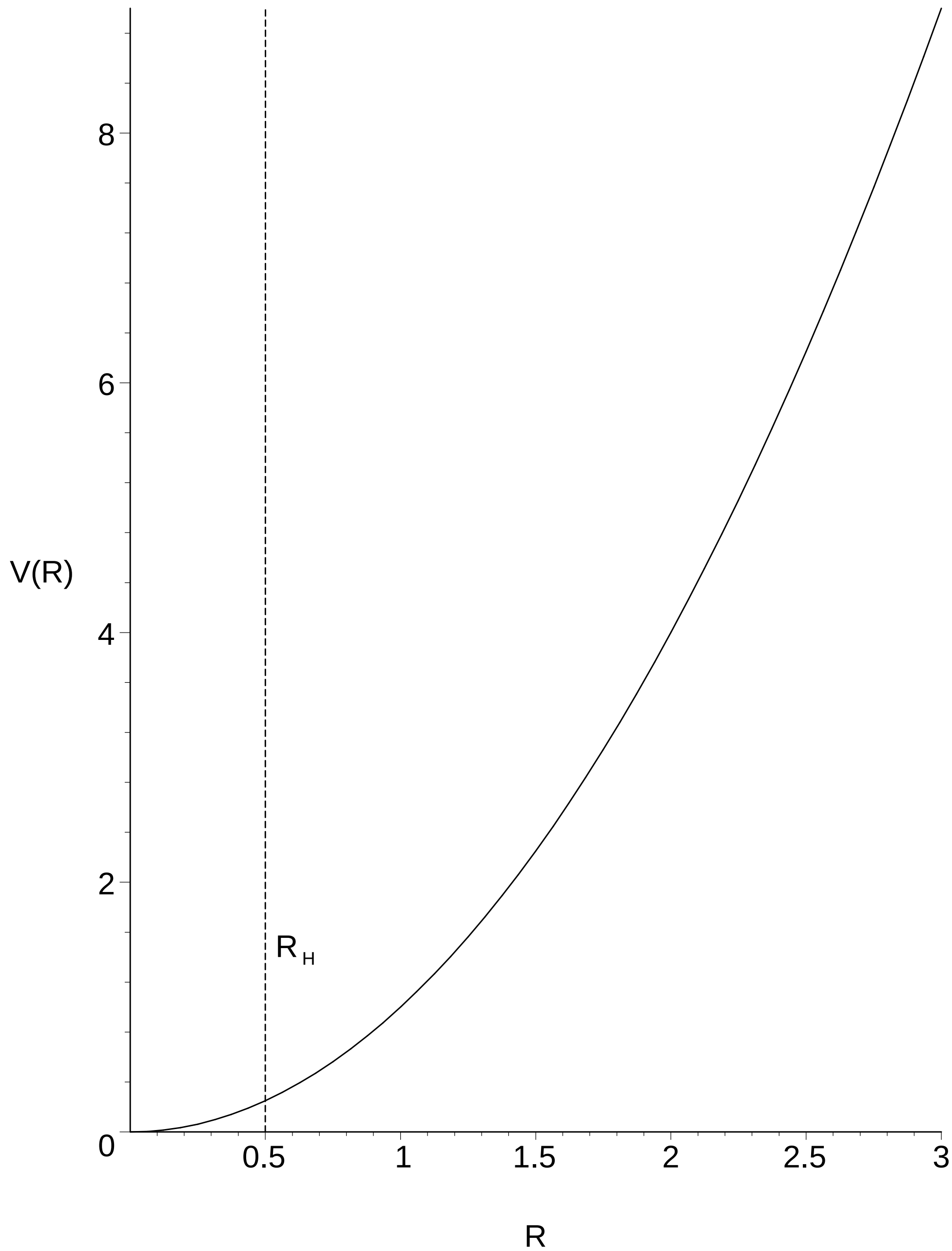

Figure 1: The harmonic oscillator potential for our problem with $M_{\rm Pl}\omega_0^2/2 = 1$. The dashed line shows a typical position of the classical horizon radius, R_H.

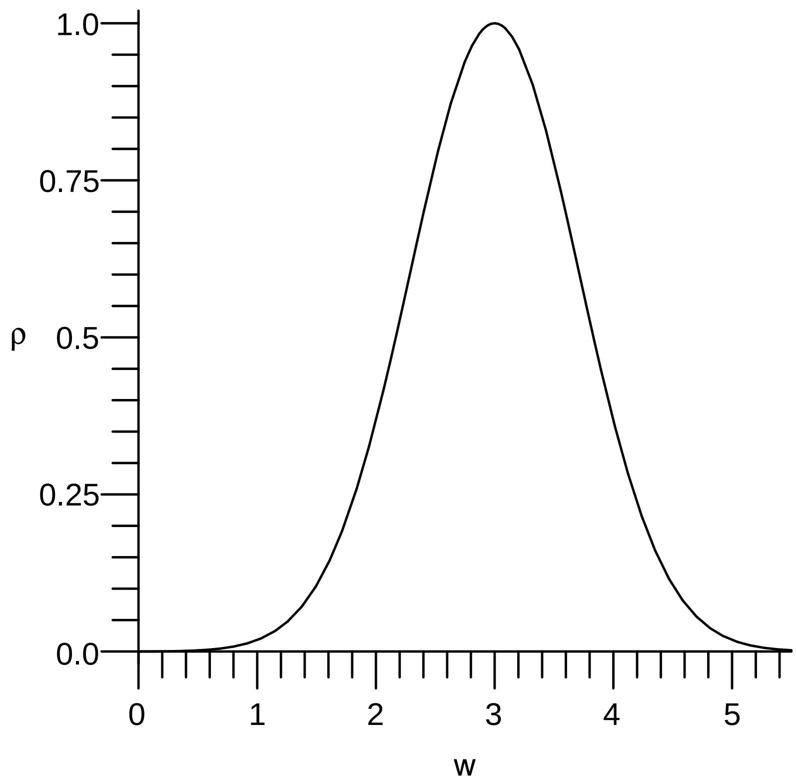

Figure 2: The probability density for the wave packet given in Eq. (47) for $T = 0$ (the initial state).

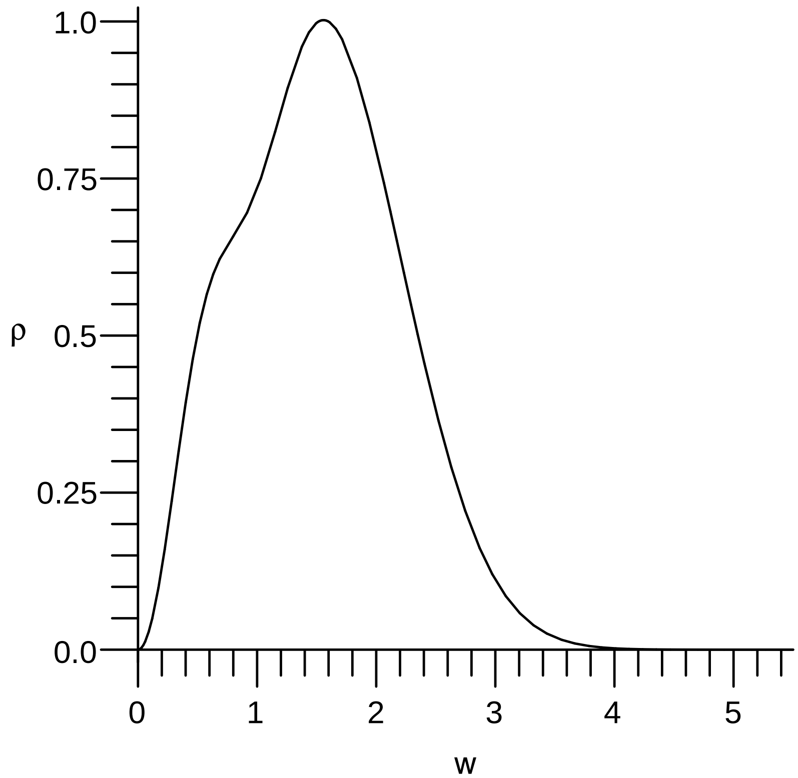

Figure 3: The probability density for the wave packet given in Eq. (47) for $T = 1.04$.

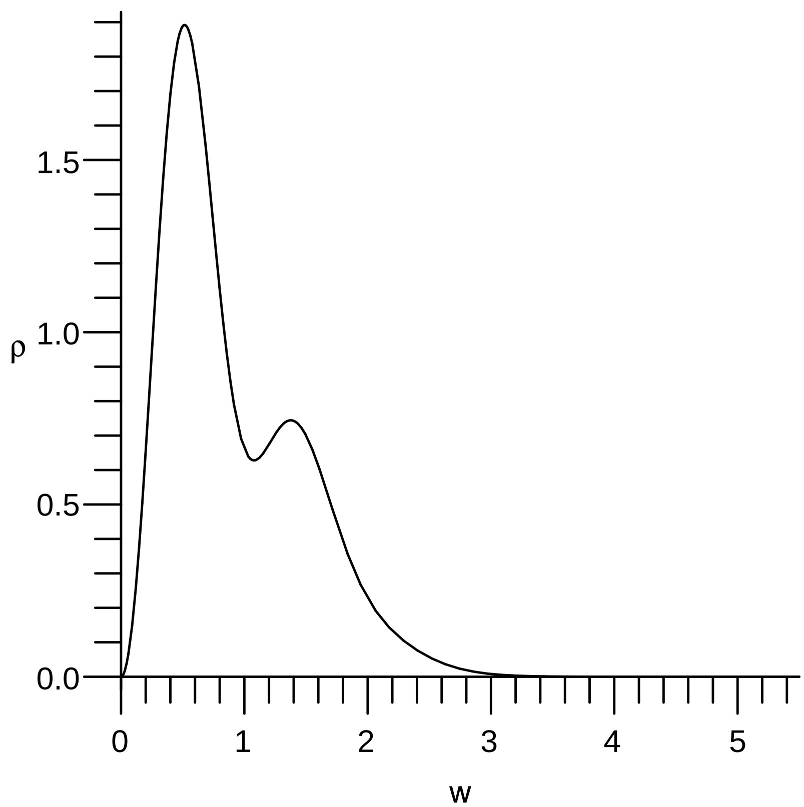

Figure 4: The probability density for the wave packet given in Eq. (47) for $T = 1.3$.

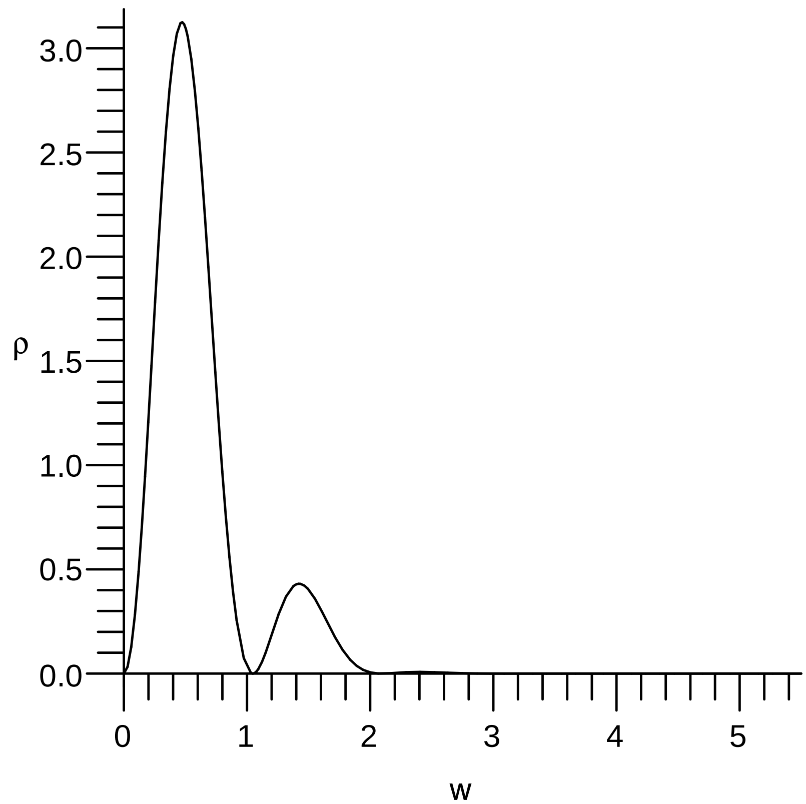

Figure 5: The probability density for the wave packet given in Eq. (47) for $T = 1.57, \approx \pi/2$.

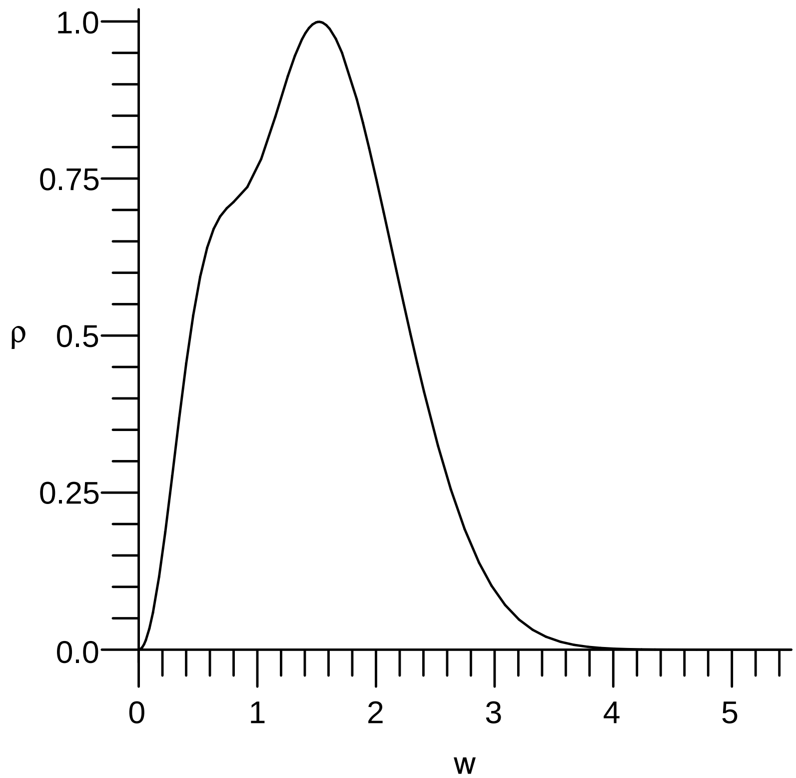

Figure 6: The probability density for the wave packet given in Eq. (47) for $T = 2.08$.

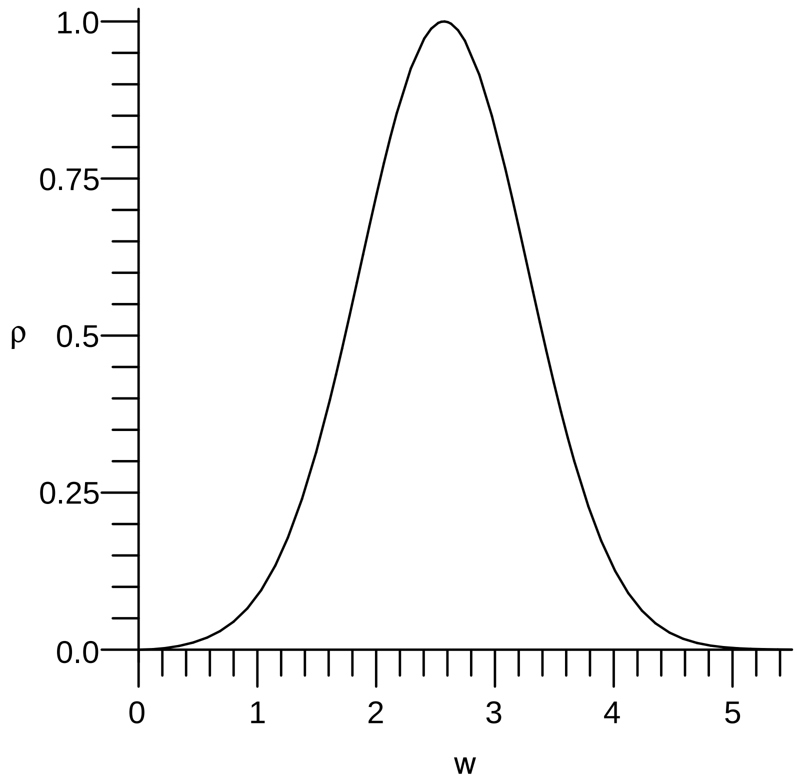

Figure 7: The probability density for the wave packet given in Eq. (47) for $T = 2.6$.

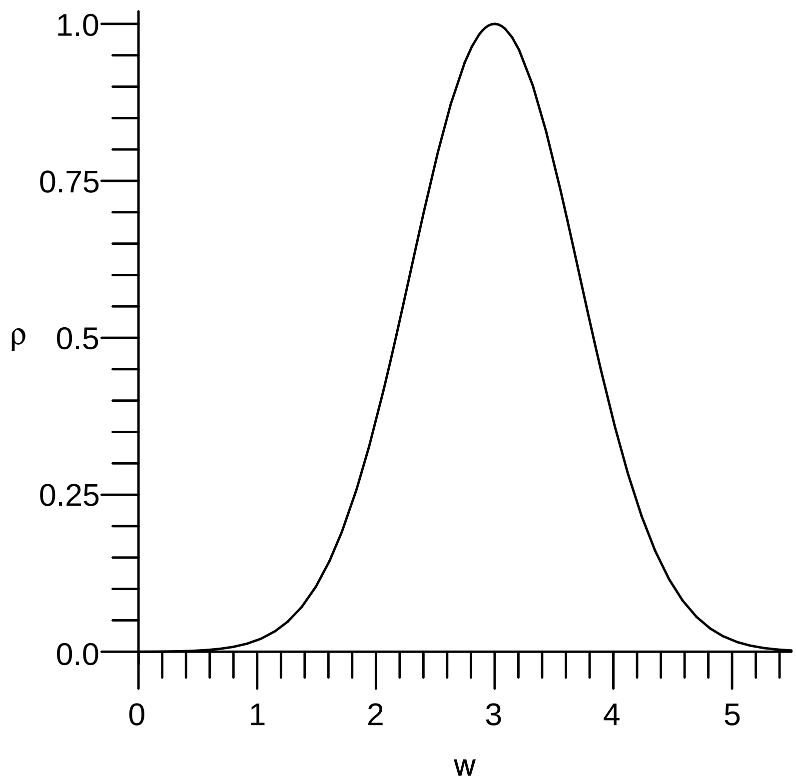

Figure 8: The probability density for the wave packet given in Eq. (47) for $T = 3.14, \approx \pi$.